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Asymptotic normality of the size of the giant component via a random walk

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ABSTRACT

In this paper we give a simple new proof of a result of Pittel and Wormald concerning the asymptotic value and (suitably rescaled) limiting distribution of the number of vertices in the giant component of $G(n, p)$ above the scaling window of the phase transition. Nachmias and Peres used martingale arguments to study Karp's exploration process, obtaining a simple proof of a weak form of this result. We use slightly different martingale arguments to obtain a much sharper result with little extra work.

1. Introduction and results

The component of a random graph containing a given vertex may be 'explored' by a step-by-step process that is by now well known, described in detail below. A key feature of this process is that vertices are 'examined' one at a time, and tested for edges to 'new' vertices. This means that the behavior of the exploration is closely connected to that of a certain random walk. In the context of random graphs, this process was introduced by Karp [4] in 1990; slightly earlier, Martin-Löf [5] used essentially the same process in a different context, namely the study of epidemics, where it arises even more naturally. Somewhat later, Aldous [1] introduced a variant of the process adapted to explore *all* components of a random graph; recently, analyzing this latter exploration with martingale techniques related to those in [5], Nachmias and Peres [6] gave a simple proof that in the weakly supercritical range, i.e., when $p = (1 + \varepsilon)/n$ where $\varepsilon = \varepsilon(n)$ satisfies $\varepsilon \to 0$ but $\varepsilon^3 n \to \infty$, the largest

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component of $G(n,p)$ contains $2\varepsilon n + o_{\mathrm{p}}(\varepsilon n)$ vertices. (They also studied the weakly subcritical case, which we shall not discuss further here.)

Here we shall analyze the same process more carefully, obtaining a simple new proof of the following asymptotic normality result due to Pittel and Wormald [8]. Let $\rho = \rho_\lambda$ denote the survival probability of the Galton–Watson branching process in which the number of offspring of each individual has a Poisson distribution with mean λ. For $\lambda > 1$ we may write ρ_λ as the unique positive solution to

$$1 - \rho = e^{-\lambda\rho}. \tag{1}$$

When $\lambda > 1$ we write λ_* for $\lambda(1-\rho_\lambda)$; this is often known as the *dual branching process parameter* to λ, and satisfies $\lambda_* < 1$ and $\lambda_* e^{-\lambda_*} = \lambda e^{-\lambda}$. (The corresponding Poisson branching process provides an approximation of the random graph in the vicinity of a generic vertex outside the giant component.)

Theorem 1. *Let $p = \lambda/n$ where $\lambda = \lambda(n)$ satisfies $\lambda = O(1)$ and $(\lambda-1)^3 n \to \infty$ as $n \to \infty$, and let L_1 denote the number of vertices in the largest component of $G(n,p)$. Then*

$$\frac{L_1 - \rho n}{\sigma} \xrightarrow{\mathrm{d}} N(0,1),$$

where $\xrightarrow{\mathrm{d}}$ denotes convergence in distribution, $N(0,1)$ is the standard normal distribution, $\rho = \rho_\lambda > 0$ is defined by (1)*, and*

$$\sigma^2 = \frac{\rho(1-\rho)}{(1-\lambda_*)^2} n.$$

The special case of this result in which λ is constant goes back to Stepanov [9] (see also Pittel [7]); the form above is due to Pittel and Wormald [8], who proved much more, including asymptotic joint normality of the sizes of the largest component and of its 2-core.

Specializing to the barely supercritical case, the formulae above simplify considerably. Indeed, it is easy to check that if $\lambda = 1+\varepsilon$ and $\varepsilon \to 0$, then $\rho_\lambda = 2\varepsilon + O(\varepsilon^2)$, and $\lambda_* = 1 - \varepsilon + O(\varepsilon^2)$. Thus Theorem 1 has the following corollary.

Corollary 2. *Let $\varepsilon = \varepsilon(n)$ satisfy $\varepsilon \to 0$ and $\varepsilon^3 n \to \infty$, and let L_1 denote the number of vertices in the largest component of $G(n,(1+\varepsilon)/n)$. Then*

$$\frac{L_1 - \rho n}{\sqrt{2\varepsilon^{-1} n}} \xrightarrow{\mathrm{d}} N(0,1), \tag{2}$$

where $\rho > 0$ is defined by (1) *with $\lambda = 1+\varepsilon$.* □

Under the conditions of Corollary 2 we have $\rho \sim 2\varepsilon$, while the standard deviation $\sqrt{2\varepsilon^{-1}n}$ is $o(\varepsilon n)$, so Corollary 2 implies in particular the result of Nachmias and Peres [6] mentioned earlier.

2. The proof

We consider the component exploration process as in [6], itself based on those of Karp [4], Martin-Löf [5] and Aldous [1], although we shall use slightly different terminology and initial conditions. At each step, every vertex will have one of three states, *active*, *explored*, or *unseen*. The exploration will take place in n steps, at times $t = 1, \ldots, n$, starting from the initial state at time 0, when every vertex is unseen.

Fix an order on the vertices. At step $1 \leqslant t \leqslant n$ (i.e., going from time $t-1$ to time t) let v_t be the first active vertex, if there are any; otherwise v_t is the first unseen vertex. In the latter case we say that we 'start a new component' at step t. Having defined v_t, reveal all edges from v_t to (other) unseen vertices; let η_t be the number of such edges, and label the corresponding neighbors of v_t as active; label v_t itself as explored. After t steps of the process, exactly t vertices have been explored. We write A_t and U_t for the numbers of active and unseen vertices after $0 \leqslant t \leqslant n$ steps, so $U_t = n - t - A_t$, $A_0 = 0$ and $U_0 = n$.

After n steps, it is very easy to see that the process has revealed a spanning forest in G, having first revealed a spanning tree of one component, then a spanning tree of another component (if there is more than one), and so on.

Write C_t for the number of components started by time t, and set $X_t = A_t - C_t$. We claim that

$$X_t = A_t - C_t = \sum_{i=1}^{t} (\eta_i - 1). \tag{3}$$

Indeed, if in step t we do not start a new component, then we explore an active vertex and then change η_t vertices from unseen to active, so $A_t - A_{t-1} = \eta_t - 1$ and $C_t = C_{t-1}$. If we do start a new component, which happens if and only if $A_{t-1} = 0$, then we explore an unseen vertex, so $A_t - A_{t-1} = A_t = \eta_t$ and $C_t - C_{t-1} = 1$. This establishes (3).

Let $0 = t_0 < t_1 < t_2 < \cdots < t_k = n$ enumerate $\{t\colon\ A_t = 0\}$, i.e., the set of times at which there are no active vertices. We start exploring the ith component at time $t_{i-1} + 1$ and finish at time t_i, so

$$L_1 = \max\{t_i - t_{i-1}\colon\ 1 \leqslant i \leqslant k\}. \tag{4}$$

Since $C_t = i$ for $t_{i-1} < t \leqslant t_i$, recalling that $X_t = A_t - C_t$ we have

$$t_i = \inf\{t\colon\ X_t = -i\}. \tag{5}$$

Writing $c(G)$ for the number of components of $G = G(n, p)$, note that $X_n = -c(G)$, and that X_t may decrease by at most one at each step, so the infimum is defined for all $1 \leqslant i \leqslant c(G)$.

Let $\mathcal{F}_t$ denote the sigma-field generated by $\eta_1, \ldots, \eta_t$; in other words, $\mathcal{F}_t$ is the (finite, of course) sigma-field generated by all information revealed by step t. Set $U'_t = U_t$ if $A_t > 0$ and $U'_t = U_t - 1$ otherwise. Then U'_t is the number of edges tested at step $t + 1$. Hence, given $\mathcal{F}_t$, the random variable η_{t+1} has a binomial distribution with parameters U'_t and p:

$$\mathbb{P}(\eta_{t+1} = k \mid \mathcal{F}_t) = \binom{U'_t}{k} p^k (1-p)^{U'_t - k}.$$

If we know the sequence (η_t), then we know the entire outcome of the process, and in particular L_1. More precisely, we can use (3) to find (X_t), then (5) to find the t_i (and thus (C_t), (A_t) and (U_t)), and finally (4) gives us L_1.

So far we have been following (with minor modifications) the definitions and initial analysis in [6]. But now our analysis takes a different route.

Let us write D_t for the expectation of $\eta_t - 1$ given $\mathcal{F}_{t-1}$, noting that D_t is random, and satisfies

$$D_{t+1} = \mathbb{E}(\eta_{t+1} - 1 \mid \mathcal{F}_t) = pU'_t - 1.$$

Recalling that $U_t = n - t - A_t = n - t - X_t - C_t$, and noting that $U'_t = U_t - (C_{t+1} - C_t)$, this gives

$$D_{t+1} = p(n - t - X_t - C_{t+1}) - 1. \tag{6}$$

Our next aim is to approximate the process (X_t) that we wish to study by a simpler process $(\widetilde{X}_t)$, consisting of a deterministic term plus a term closely related to a martingale. Let $\Delta_t = \eta_t - 1 - D_t$, so $\mathbb{E}(\Delta_t \mid \mathcal{F}_{t-1}) = 0$ by the definition of D_t. From (3), (6) and $\eta_{t+1} - 1 = D_{t+1} + \Delta_{t+1}$ we obtain the recurrence

$$X_{t+1} = (1-p)X_t + \Delta_{t+1} + p(n-t) - 1 - pC_{t+1}. \tag{7}$$

Let

$$x_t = n - t - n(1-p)^t,$$

so $x_0 = 0$ and

$$x_{t+1} = (1-p)x_t + p(n-t) - 1. \tag{8}$$

Subtracting (8) from (7) we see that

$$X_{t+1} - x_{t+1} = (1-p)(X_t - x_t) + \Delta_{t+1} - pC_{t+1},$$

whence

$$X_t - x_t = \sum_{i=1}^{t} (1-p)^{t-i}(\Delta_i - pC_i). \tag{9}$$

With this in mind, we define our approximating process by

$$\widetilde{X}_t = x_t + \sum_{i=1}^{t} (1-p)^{t-i}\Delta_i. \tag{10}$$

Lemma 3. *For any $p > 0$ and any $1 \leqslant t \leqslant n$ we have*

$$|X_t - \widetilde{X}_t| \leqslant ptC_t.$$

Proof. From (9) and (10) we have

$$X_t - \widetilde{X}_t = -\sum_{i=1}^{t} (1-p)^{t-i} pC_i.$$

The result follows immediately since there are t terms in the sum, each bounded by pC_t. □

Let

$$S_t = \sum_{i=1}^{t} (1-p)^{-i}\Delta_i,$$

so (S_t) is a martingale, and

$$\widetilde{X}_t = x_t + (1-p)^t S_t. \tag{11}$$

As we shall see below, it is easy to obtain very precise results about the distribution of $(\widetilde{X}_t)$; before turning to the details, let us indicate in rather vague terms why this should be the case.

The variance of each Δ_i is $O(1)$, so S_t and hence $(1-p)^t S_t$ have variance $O(t)$ and size $O_{\mathrm{p}}(\sqrt{t})$. It is true that the distribution of Δ_t depends on earlier values of X_i in a way that is hard to evaluate exactly, but the dependence is weak: the conditional variance of Δ_t is simply $p(1-p)U'_{t-1}$, so if we can bound the earlier X_i within an additive error of $o(n)$, then we obtain a bound on the variance of Δ_t accurate to within a factor $1 + o(1)$. This gives only an $o_{\mathrm{p}}(\sqrt{t})$ additive error in the martingale term, which is negligible compared to the random variation. (It will turn out that we hit the giant component before seeing many other components, so the additional ptC_t error from Lemma 3 will be negligible.) This strongly suggests that given that Theorem 1 is true, there should be a simple proof based on the analysis of $(\widetilde{X}_t)$. As we shall see, this is indeed the case.

From now on we assume that $p = \lambda/n$, where $\lambda = \lambda(n) > 1$ is bounded. More explicitly, we assume $\lambda < M$ for some constant M. Often, we write $\lambda = 1 + \varepsilon$; we assume also that $\varepsilon^3 n \to \infty$.

For the moment, we study $(\widetilde{X}_t)$. Let us first start with a standard observation; the second part is a special case of Doob's maximal inequality [3, Ch. III, Theorem 2.1].

Lemma 4. *Let $(Z_t)_0^\infty$ be a discrete-time martingale with filtration $(\mathcal{F}_t)$ and mean $Z_0 = 0$. Write I_t for the increment $Z_t - Z_{t-1}$. Then*

$$\mathrm{Var}(Z_t) = \sum_{i=1}^{t} \mathrm{Var}(I_i) = \sum_{i=1}^{t} \mathbb{E}\bigl(\mathrm{Var}(I_i \mid \mathcal{F}_{i-1})\bigr), \tag{12}$$

and for any $M \geqslant 0$,

$$\mathbb{P}\Bigl(\max_{i \leqslant t} |Z_i| \geqslant M\Bigr) \leqslant \mathrm{Var}(Z_t)/M^2.$$

Proof. For the first statement, observe that $\mathbb{E}I_i = 0$ for all i and $\mathbb{E}Z_t = 0$, while for $i < j$ we have $\mathbb{E}(I_i I_j) = \mathbb{E}(\mathbb{E}(I_i I_j \mid \mathcal{F}_{j-1})) = \mathbb{E}(0) = 0$. Hence $\mathrm{Var}(Z_t) = \mathbb{E}Z_t^2 = \mathbb{E}(\sum_1^t I_i)^2 = \sum_i \mathbb{E}I_i^2 = \sum_i \mathrm{Var}(I_i)$. Also, $\mathbb{E}(\mathrm{Var}(I_i \mid \mathcal{F}_{i-1})) = \mathbb{E}(\mathbb{E}(I_i^2 \mid \mathcal{F}_{i-1})) = \mathbb{E}I_i^2$, proving (12).

For the second statement, apply Doob's maximal inequality. Alternatively, simply modify the martingale if $|Z_i| \geqslant M$ holds for any i: let T be the (random) first such i, or $T = t$ if there is no such i, and set $Z'_j = Z_j$ for $j \leqslant T$ and $Z'_j = Z_T$ for $j > T$. Since T is a stopping time, the conditional distribution of $I'_i = Z'_i - Z'_{i-1}$ given $\mathcal{F}_{i-1}$ is either the same as that of I_i, or zero, so the conditional variances of the I'_i are at most those of the I_i. Hence, by (12), $\mathrm{Var}(Z'_t) \leqslant \mathrm{Var}(Z_t)$. Since $\max_{i \leqslant t} |Z_i| \geqslant M$ if and only if $|Z'_t| \geqslant M$, applying Chebyshev's inequality gives the result. □

Let us write $\mathrm{CBi}(m, p)$ for the *centered binomial distribution* obtained by subtracting mp from a random variable with binomial distribution $\mathrm{Bi}(m, p)$. Note that the variance of this distribution is $mp(1-p)$. The conditional distribution of Δ_t given $\mathcal{F}_{t-1}$ is exactly that of a centered binomial $\mathrm{CBi}(U'_{t-1}, p)$. (Previously, we first subtracted one, and then centered, but of course this is the same as centering directly.) It follows that the differences $I_i = S_i - S_{i-1} = (1-p)^{-i}\Delta_i$ satisfy

$$\mathrm{Var}(I_i \mid \mathcal{F}_{i-1}) = (1-p)^{-2i} U'_{i-1} p(1-p), \tag{13}$$

so

$$\mathrm{Var}(I_i \mid \mathcal{F}_{i-1}) \leqslant (1-p)^{-2n} np \leqslant (1 - M/n)^{-2n} M = O(1).$$

For any (deterministic) function $t = t(n)$, Lemma 4 thus gives

$$\sup_{i \leqslant t} |S_i| = O_{\mathrm{p}}(\sqrt{t}). \tag{14}$$

Let $f(t) = f_n(t) = n - t - ne^{-pt}$ be the continuous-time form of the idealized trajectory of $(\widetilde{X}_t)$ (and hence of (X_t)). It is easy to check that $|f(t) - x_t| = O(1)$, uniformly in $p \leqslant M/n$ and $0 \leqslant t \leqslant n$; our next lemma shows that $(\widetilde{X}_t)$ remains close to $f_n(t)$.

Lemma 5. *For any $1 \leqslant t = t(n) \leqslant n$ we have*

$$\sup_{i \leqslant t} \bigl|\widetilde{X}_t - f_n(t)\bigr| = O_{\mathrm{p}}(\sqrt{t}).$$

Proof. Immediate from (14), (11) and $|f_n(t) - x_t| = O(1)$. □

Together, Lemmas 3 and 5 show that (X_t) remains close to the idealized trajectory $f(t)$, as long as C_t is not too large. As in [6], the basic idea is now to consider the solution $t_1 = \rho n$ to $f(t_1) = 0$, and choose a suitable t_0. We shall show that in the interval $[t_0, t_1 - t_0]$ the function $f(t)$ is far enough away from zero that X_t remains positive, so no new component is started in this interval. Then we consider more precisely the time when X_t crosses below its previous minimum level and use (5) to obtain Theorem 1.

We start by examining f. Note that

$$f'(t) = -1 + npe^{-pt} = p\bigl(n - t - f(t)\bigr) - 1, \tag{15}$$

and that $f''(t) = -np^2e^{-pt}$ is negative and uniformly bounded by M^2/n. Since $f'(0) = np - 1 = \varepsilon$, it follows that if $t \leqslant \varepsilon n/(2M^2)$, then $f'(t) \geqslant \varepsilon/2$ and, integrating, that

$$f(t) \geqslant \varepsilon t/2. \tag{16}$$

From now on let us pick a function $\omega = \omega(n)$ tending to infinity slowly, in particular with $\omega^6 = o(\varepsilon^3 n)$. Set

$$\sigma_0 = \sqrt{\varepsilon n}$$

and

$$t_0 = \omega\sigma_0/\varepsilon,$$

ignoring, as usual, the irrelevant rounding to integers. Note for later that $t_0 = o(\varepsilon n)$.

Lemma 6. *Let $Z = -\inf\{X_t\colon\ t \leqslant t_0\}$ denote the number of components completely explored by time t_0, and let $T_0 = \inf\{t\colon\ X_t = -Z\}$ be the time at which we finish exploring the last such component. Then $Z \leqslant \sigma_0/\omega$ and $T_0 \leqslant \sigma_0/(\varepsilon\omega)$ hold whp.*

Considering the initial trajectory of the process (X_t), it is not hard to check that in fact $Z = O_{\mathrm{p}}(\varepsilon^{-1})$ and $T_0 = O_{\mathrm{p}}(\varepsilon^{-2})$, but the weaker bounds above suffice.

Proof. Let $k = \sigma_0/\omega$. Note that by choice of ω we have $k/\sqrt{t_0} \to \infty$. Let $\mathcal{A}$ denote the event that $\sup_{t\leqslant t_0} |\widetilde{X}_t - f(t)| < k/2$. Then by Lemma 5, $\mathcal{A}$ holds whp.

At time T_0 we have $X_{T_0} = -Z$. Noting that $pt_0 = o(1)$, we have $pt_0 \leqslant 1/2$ if n is large enough, which we assume from now on. Since $T_0 \leqslant t_0$ by definition, it follows that $pT_0 \leqslant 1/2$. But then Lemma 3 gives

$$|X_{T_0} - \widetilde{X}_{T_0}| \leqslant pT_0C_{T_0} \leqslant Z/2,$$

and thus $\widetilde{X}_{T_0} \leqslant -Z/2$. Since $f(t) \geqslant 0$ for $t \leqslant t_0 < \rho n$, this gives $|\widetilde{X}_{T_0} - f(T_0)| \geqslant Z/2$. Hence, whenever $\mathcal{A}$ holds, we have $Z \leqslant k$, and the first statement follows.

Turning to the second statement, recall from (16) that $f(t) \geqslant \varepsilon t/2$ for $t \leqslant t_0 = o(\varepsilon n)$. Consider the interval $I = [\sigma_0/(\varepsilon\omega), t_0]$. In this interval we have $f(t) \geqslant \sigma_0/(2\omega) = k/2$, so if $\mathcal{A}$ holds then $\widetilde{X}_t > 0$ for all $t \in I$. As shown above, we have $\widetilde{X}_{T_0} \leqslant -Z/2 \leqslant 0$, so whenever $\mathcal{A}$ holds then $T_0 \notin I$. Since $T_0 \leqslant t_0$ by definition, this completes the proof. $\square$

Let $T_1 = \inf\{t\colon\ X_t = -Z - 1\}$. Then by the properties of the exploration process, there is a component with $T_1 - T_0$ vertices; we aim to show that this component has size close to the anticipated size of the giant component.

Since $np = O(1)$, by Lemmas 3 and 6 we have that

$$\sup_{t\leqslant T_1} |X_t - \widetilde{X}_t| \leqslant \sigma_0/\sqrt{\omega} \tag{17}$$

holds whp.

Let $t_1 = \rho n$, noting that $t_1 \sim 2\varepsilon n$ if $\varepsilon \to 0$, and that t_1 is the unique positive solution to $f(t) = 0$. Let $t_1^- = t_1 - t_0$ and $t_1^+ = t_1 + t_0$. Note that $t_1^+ = O(\varepsilon n) = O(\sigma_0^2)$. From (17) and Lemma 5 we have that

$$\sup_{t\leqslant \min\{T_1, t_1^+\}} \bigl|X_t - f(t)\bigr| \leqslant \sqrt{\omega}\sigma_0 \tag{18}$$

holds whp.

Let $a = -f'(t_1)$, so from (15) and the definition of t_1 we have

$$a = -f'(t_1) = 1 - p(n - t_1) = 1 - \lambda(1 - \rho) = 1 - \lambda_*,$$

where λ_* is the dual branching process parameter to λ. In particular, $a = \Theta(\varepsilon)$. Since $f(t_1) = 0$ and $f''(t)$ is uniformly $O(1/n)$, recalling that $t_0 = o(\varepsilon n)$ it follows easily that $f(t_1^-)$ and $f(t_1^+)$ are both of order $\varepsilon t_0 = \omega\sigma_0$. To be concrete, if n is large enough, then we certainly have

$$f\big(t_1^-\big) \geqslant 10\sqrt{\omega}\sigma_0 \quad \text{and} \quad f\big(t_1^+\big) \leqslant -10\sqrt{\omega}\sigma_0,$$

say. Since $f(t_0) \geqslant \varepsilon t_0/2 \geqslant 10\sqrt{\omega}\sigma_0$ and f is unimodal, we have $\inf_{t_0 \leqslant t \leqslant t_1^-} f(t) \geqslant 10\sqrt{\omega}\sigma_0$. Let $\mathcal{B}$ denote the event described in (18). Then, whenever $\mathcal{B}$ holds, we have $X_t \geqslant 0$ for $t_0 \leqslant t \leqslant \min\{T_1, t_1^-\}$. Since $X_{T_1} \leqslant -Z - 1 < 0$, this implies $T_1 > t_1^-$.

Recall from Lemma 6 that (crudely) $Z \leqslant \sigma_0$ whp. Suppose $Z \leqslant \sigma_0$, $\mathcal{B}$ holds, and $T_1 > t_1^+$. Then from $\mathcal{B}$ and the bound on $f(t_1^+)$ we have $X_{t_1^+} \leqslant -9\sqrt{\omega}\sigma_0 < -Z$, contradicting $T_1 > t_1^+$. It follows that $T_1 \leqslant t_1^+$ holds whp.

At this point we have shown that $|T_1 - t_1| \leqslant t_0$ holds whp, which gives $|T_1 - T_0 - t_1| \leqslant 2t_0$. Since ω may tend to infinity arbitrarily slowly, this already shows that $T_1 - T_0 = t_1 + O_{\mathrm{p}}(\sigma_0/\varepsilon) = \rho n + O_{\mathrm{p}}(\sqrt{\varepsilon^{-1}n})$. To go further, we next analyze the distribution of X_{t_1} more precisely.

From Lemma 6 and the bound $T_1 > t_1^-$ whp just proved, whp we have $C_{t_1^-} = Z \leqslant \sigma_0/\omega$. Noting that $t_0 = t_1 - t_1^- = o(n)$, it follows that $\mathbb{E}C_{t_1} = o(n)$. Lemma 3 and Lemma 5 thus give $|X_t - f(t)| = o_{\mathrm{p}}(n)$, uniformly in $t \leqslant t_1$. Since $X_t - f(t)$ is deterministically bounded by n, it follows that $\mathbb{E}|X_t - f(t)|$ and hence $\mathbb{E}|X_t + C_{t+1} - f(t)|$ are $o(n)$, uniformly in $t \leqslant t_1$. Let $u_t = n - t - f(t) = ne^{-pt}$. Since $U_t' = n - t - (X_t + C_{t+1})$, we have shown that

$$\mathbb{E}\sum_{t=0}^{t_1-1} \big|U_t' - u_t\big| = o(t_1 n) = o\big(\varepsilon n^2\big). \tag{19}$$

Note that

$$\begin{aligned} p(1-p)\sum_{t=0}^{t_1-1}(1-p)^{-2t}u_t &\sim p\sum_{t=0}^{t_1-1} e^{2pt}ne^{-pt} \\ &\sim n^2 p\int_0^{\rho} e^{\lambda x}\,\mathrm{d}x = n\lambda\lambda^{-1}\big(e^{\lambda\rho} - 1\big) = n\rho/(1-\rho), \end{aligned} \tag{20}$$

using $e^{-\lambda\rho} = 1 - \rho$ in the last step.

Lemma 7. *The distribution of S_{t_1} is asymptotically normal with mean 0 and variance $n\rho/(1-\rho)$.*

Proof. Recall that (S_t) is a martingale with $S_0 = 0$, and that the conditional distribution of the ith difference $(1-p)^{-i}\Delta_i$ is $(1-p)^{-i}$ times a centered binomial $\mathrm{CBi}(U_{i-1}', p)$, and has conditional variance given by (13). The result follows easily by a standard martingale central limit theorem such as Theorem 2 of Brown [2]. Note that here the differences are not uniformly bounded. However, we can write Δ_i as the sum of a random number U_{i-1}' of $\mathrm{CBi}(1, p)$ random variables, plus $n - U_{i-1}'$ zero variables. We can take the new variables multiplied by $(1-p)^{-i}$ as the differences of a martingale (S_j') with the property that $S_t = S_{nt}'$. In this way we obtain a martingale with the same (random) final value in which the differences are bounded by $(1-p)^{-n} = O(1)$. The (random) sum of the (old or new) conditional variances is exactly $s = \sum_{t=0}^{t_1-1}(1-p)^{-2t}U_{t-1}'p(1-p)$. By (19) and (20) the ratio of s to $n\rho/(1-\rho)$ converges to 1 in probability, as required for the martingale central limit theorem. $\square$

To relate the distribution of T_1 to that of X_{t_1} (or $\widetilde{X}_{t_1}$) we use the fact that (X_t) has slope approximately $-a$ near t_1; a similar argument was given by Martin-Löf [5].

Lemma 8. *We have*

$$\sup_{|t-t_1|\leqslant t_0} \left|\widetilde{X}_t - \widetilde{X}_{t_1} - a(t_1 - t)\right| = o_{\mathrm{p}}(\sigma_0).$$

Proof. From (11) we may write $\widetilde{X}_t - \widetilde{X}_{t_1}$ as

$$x_t - x_{t_1} + (1-p)^t S_t - (1-p)^{t_1} S_{t_1} = \bigl(f(t) - f(t_1)\bigr) + (1-p)^t S_t - (1-p)^{t_1} S_{t_1} + O(1).$$

Recalling that $f'(t_1) = -a$ and $f''(t) = O(1/n)$ uniformly in t, the difference between the first term and $a(t_1 - t)$ is $O(|t-t_1|^2/n) = O(t_0^2/n) = o(\sigma_0)$. For the rest, note that

$$\left|(1-p)^t - (1-p)^{t_1}\right| \leqslant \left|1 - (1-p)^{|t-t_1|}\right| \leqslant p|t-t_1| \leqslant pt_0.$$

Since $S_{t_1} = O_{\mathrm{p}}(\sqrt{t_1})$ and $pt_0\sqrt{t_1} = O(n^{-1}\omega\sigma_0\varepsilon^{-1}\sqrt{\varepsilon n}) = o(\sigma_0)$, it thus suffices to show that $\sup_{|t-t_1|\leqslant t_0} |S_t - S_{t_1}| = o_{\mathrm{p}}(\sigma_0)$. But this follows easily by applying Lemma 4 to the martingale $(S_t - S_{t_1^-})_{t=t_1^-}^{t_1^+}$, which has final variance $O(t_0) = o(\sigma_0^2)$. □

Proof of Theorem 1. Recall from Lemma 6 that Z, the number of components explored by time t_0, satisfies $Z = o_{\mathrm{p}}(\sigma_0)$. We have shown above that whp $T_1 = \inf\{t\colon X_t = -Z-1\}$ lies between t_1^- and t_1^+. From (17), X_t is within $o_{\mathrm{p}}(\sigma_0)$ of $\widetilde{X}_t$ at least until T_1. It follows that at time T_1, we have $\widetilde{X}_t = o_{\mathrm{p}}(\sigma_0)$. Since $a = \Theta(\varepsilon)$, Lemma 8 thus gives

$$T_1 = t_1 + \widetilde{X}_{t_1}/a + o_{\mathrm{p}}(\sigma_0/\varepsilon). \tag{21}$$

From Lemma 7, (11) and the fact that $f(t_1) = 0$, we have that $\widetilde{X}_{t_1}$ is asymptotically normal with mean 0 and variance

$$(1-p)^{2t_1} n\rho/(1-\rho) \sim e^{-2\lambda\rho} n\rho/(1-\rho) = n\rho(1-\rho).$$

Hence $\widetilde{X}_{t_1}/a$ is asymptotically normal with mean 0 and variance

$$n\rho(1-\rho)/a^2 = \sigma^2.$$

Since this variance is of order $\varepsilon^{-1}n = \varepsilon^{-2}\sigma_0^2$, the $o_{\mathrm{p}}(\sigma_0/\varepsilon)$ error term in (21) is irrelevant, and T_1 is asymptotically normal with mean $t_1 = \rho n$ and variance σ^2. Finally, from Lemma 6 we have $T_0 = o_{\mathrm{p}}(\sigma_0/\varepsilon)$. It follows that $T_1 - T_0$ is asymptotically normal with the parameters claimed in the theorem.

This shows the existence of a component with the claimed size. As shown by Nachmias and Peres [6], it is easy to check that the rest of the graph corresponds to a subcritical random graph, and whp will not contain a larger component. □

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